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Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling

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ABSTRACT. The normal inverse Gaussian distribution is defined as a variance-mean mixture of a normal distribution with the inverse Gaussian as the mixing distribution. The distribution determines an homogeneous Lévy process, and this process is representable through subordination of Brownian motion by the inverse Gaussian process. The canonical, Lévy type, decomposition of the process is determined. As a preparation for developments in the latter part of the paper the connection of the normal inverse Gaussian distribution to the classes of generalized hyperbolic and inverse Gaussian distributions is briefly reviewed. Then a discussion is begun of the potential of the normal inverse Gaussian distribution and Lévy process for modelling and analysing statistical data, with particular reference to extensive sets of observations from turbulence and from finance. These areas of application imply a need for extending the inverse Gaussian Lévy process so as to accommodate certain, frequently observed, temporal dependence structures. Some extensions, of the stochastic volatility type, are constructed via an observation-driven approach to state space modelling. At the end of the paper generalizations to multivariate settings are indicated.

Key words: conditional heteroscedasticity, finance, generalized hyperbolic distributions, generalized inverse Gaussian distributions, Lévy process, observation-driven, state space modelling, subordination, turbulence

1. Introduction

A normal variance-mean mixture distribution, here termed the normal inverse Gaussian distribution, is used to construct stochastic processes that appear of interest for statistical modelling purposes, particularly in turbulence and finance.

In section 2 we give the definition and some properties of the univariate normal inverse Gaussian distributions. These distributions generate homogeneous Lévy processes, which are considered in some detail in section 3. In particular, an alternative representation of the processes via random time change of Brownian motion, using the inverse Gaussian process to determine time, is pointed out and the canonical decomposition of the Lévy measures and of the processes themselves are determined explicitly. Section 4 describes the connection of the normal inverse Gaussian distributions to the generalized inverse Gaussian and hyperbolic distributions, in preparation for the later part of the paper. Section 5 discusses the potential applicability of the normal inverse Gaussian distributions and Lévy processes for the modelling and analysis of statistical data, particularly from turbulence and finance. The need to model certain types of dependence structures, typically observed in those areas, is described, and this forms part of the motivation for the discussion in the remaining two sections. In section 6 an observation-driven approach to state space modelling is briefly discussed and is then specialized to a rather flexible class of stochastic volatility and conditional heteroscedasticity models, exemplified using the normal inverse Gaussian distribution. Extensions to multivariate models and processes are indicated in the final section 7. At several points the constructions draw on the representation of the normal inverse Gaussian distribution as a normal variance-mean mixture.

2. The normal inverse Gaussian distribution

We define the normal inverse Gaussian distribution as the distribution on the whole real line having density function

$$g(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q\left(\frac{x-\mu}{\delta}\right)^{-1} K_1\left\{\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right\} \exp(\beta x) \tag{2.1}$$

where

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1}\alpha \exp(\delta\sqrt{(\alpha^2-\beta^2)} - \beta\mu) \tag{2.2}$$

and

$$q(x) = \sqrt{1+x^2} \tag{2.3}$$

and where K_1 is the modified Bessel function of third order and index 1. Furthermore, α, β, μ and δ are parameters, satisfying $0 \leqslant |\beta| \leqslant \alpha$, $\mu \in R$ and $0 < \delta$. The distribution is symmetric around μ provided $\beta = 0$. We shall denote this distribution by $NIG(\alpha, \beta, \mu, \delta)$. The parameters $\bar{\alpha} = \delta\alpha$ and $\bar{\beta} = \delta\beta$ are invariant under location-scale changes of x, and expressed in terms of these the density of the normal inverse Gaussian distribution becomes

$$\bar{g}(x; \bar{\alpha}, \bar{\beta}, \mu, \delta) = (\pi\delta)^{-1}\bar{\alpha} \exp(\sqrt{(\bar{\alpha}^2-\bar{\beta}^2)} - \bar{\beta}\mu/\delta) q\left(\frac{x-\mu}{\delta}\right)^{-1} \times K_1\left\{\bar{\alpha} q\left(\frac{x-\mu}{\delta}\right)\right\} \exp(\bar{\beta}x/\delta). \tag{2.4}$$

Let $IG(\delta, \gamma)$ denote the inverse Gaussian distribution with density function

$$d(z; \delta, \gamma) = (2\pi)^{-1/2}\delta \exp(\delta\gamma) z^{-3/2} \exp\{-\tfrac{1}{2}(\delta^2 z^{-1} + \gamma^2 z)\}. \tag{2.5}$$

The mean and variance of $IG(\delta, \gamma)$ are $Ez = \delta/\gamma$ and $Vz = \delta/\gamma^3$.

The $NIG(\alpha, \beta, \mu, \delta)$ distribution is a normal variance-mean mixture (Barndorff-Nielsen 1977, 1978). In fact, it occurs as the marginal distribution of x for a pair of random variables (z, x) where z follows the $IG(\delta, \sqrt{(\alpha^2-\beta^2)})$ distribution while conditional on z the distribution of x is normal with mean $\mu + \beta z$ and variance z. This is the reason why we refer to the distribution (2.1) as the normal inverse Gaussian distribution.

For fixed values of α, μ and δ the class of normal inverse Gaussian distributions constitutes an exponential model with β as canonical parameter and x as canonical statistic. The moment generating function $M(u; \alpha, \beta, \mu, \delta)$ of $NIG(\alpha, \beta, \mu, \delta)$ is therefore immediately expressible in terms of the norming constant (2.2) and we find

$$M(u; \alpha, \beta, \mu, \delta) = \exp[\delta\{\sqrt{(\alpha^2-\beta^2)} - \sqrt{(\alpha^2-(\beta+u)^2)}\} + \mu u]. \tag{2.6}$$

Thus all moments of $NIG(\alpha, \beta, \mu, \delta)$ have simple explicit expressions and, in particular, the mean and variance are

$$\kappa_1 = Ex = \mu + \delta\bar{\pi}/(1-\bar{\pi}^2)^{1/2} \tag{2.7}$$

and

$$\kappa_2 = Vx = \delta^2/\{\bar{\alpha}(1-\bar{\pi}^2)^{3/2}\} \tag{2.8}$$

where $\bar{\pi} = \bar{\beta}/\bar{\alpha} = \beta/\alpha$.

It follows, moreover, immediately from (2.6) that the normal inverse Gaussian distributions are infinitely divisible and that if $x_1, \ldots, x_m$ are independent normal inverse Gaussian random variables with common parameters α and β but having individual location-scale parameters μ_i and $\delta_i (i = 1, \ldots, m)$ then $x_+ = x_1 + \cdots + x_m$ is again distributed according to a normal inverse Gaussian law, with parameters $(\alpha, \beta, \mu_+, \delta_+)$.

We also note that the normal distribution $N(\mu, \sigma^2)$ appears as a limiting case for $\beta = 0$, $\alpha \to \infty$ and $\delta/\alpha = \sigma^2$, that the Cauchy distribution is the special case $NIG(0, 0, 1, 0)$ and that, using the well-known asymptotic formula for the Bessel function K_1

$$K_1(s) \sim \sqrt{(\pi/2)} s^{-1/2} \exp(-s) \quad \text{as } s \to \infty, \tag{2.9}$$

we have that

$$g(x; \alpha, \beta, \mu, \delta) \sim A(\alpha, \beta, \mu, \delta) q\left(\frac{x-\mu}{\delta}\right)^{-3/2} \exp[-\alpha\sqrt{\{\delta^2 + (x-\mu)^2\}} + \beta x] \tag{2.10}$$

for $\alpha > 0$ and $|x| \to \infty$ or, more generally, for $\alpha\sqrt{\{\delta^2 + (x-\mu)^2\}} \to \infty$. In (2.10) the norming constant A is given by

$$A(\alpha, \beta, \mu, \delta) = (2\pi)^{-1/2}(\alpha/\delta)^{1/2} \exp(\delta\sqrt{(\alpha^2 - \beta^2)} - \beta\mu). \tag{2.11}$$

3. Normal inverse Gaussian processes of Lévy type

We define the normal inverse Gaussian Lévy process as the homogeneous Lévy process (i.e. Lévy process with stationary increments) $\{x_t\}$ for which the moment generating function of x_t is

$$M_t(u; \alpha, \beta, \mu, \delta) = E\{\exp(ux_t)\} = M(\alpha, \beta, \mu, \delta)^t \tag{3.1}$$

where $M(\alpha, \beta, \mu, \delta)$ is given by (2.6). The moment generating function M_t of x_t is thus expressible as

$$M_t(u; \alpha, \beta, \mu, \delta) = M(u; \alpha, \beta, t\mu, t\delta). \tag{3.2}$$

3.1. Representation by subordination

As a direct consequence of the mixture representation of the normal inverse Gaussian distribution we find that the normal inverse Gaussian Lévy process x_t may be represented, via random time change of a Brownian motion, as

$$x_t = b_{z_t} + \mu t \tag{3.3}$$

where $\{b_t\}$ is the Brownian motion with drift β and diffusion coefficient 1 and where $\{z_t\}$, stochastically independent of $\{b_t\}$, is the inverse Gaussian Lévy process with parameters δ and $\sqrt{(\alpha^2 - \beta^2)}$. The latter process is defined as the homogeneous Lévy process $\{z_t\}$ for which the density of z_1 is given by (2.5). The variate z_t has the interpretation of being the first passage time to level δt of a Brownian motion with drift $\sqrt{(\alpha^2 - \beta^2)}$ and diffusion coefficient 1. In a different wording, (3.3) represents the normal inverse Gaussian Lévy process as a subordination of Brownian motion by the inverse Gaussian Lévy process.

On the other hand, the Lévy decomposition analysis given in the next subsection shows that the process can also be viewed, essentially, as a superposition of weighted independent Poisson processes, weights of all sizes occurring but those numerically small dominating the picture.

3.2. Lévy measure and Lévy decomposition

We proceed to determine the decomposition of the normal inverse Gaussian Lévy process according to the theory of homogeneous Lévy processes.

For the derivation it is inessential what the value of μ is, and we therefore assume that $\mu = 0$. Furthermore, for brevity we shall sometimes write ψ for $\alpha^2 - \beta^2$.

As above we denote the inverse Gaussian Lévy process by $\{z_t\}$. Let

$$\kappa_t(v) = \log E\{\exp(-vz_t)\}. \tag{3.4}$$

Then, by (2.5),

$$\kappa_t(v) = t\delta\{\sqrt{\psi} - \sqrt{(\psi + 2v)}\}. \tag{3.5}$$

The characteristic function of x_t, the value of the normal inverse Gaussian Lévy process at time t, may be written as

$$\varphi_t(\tau) = \exp\{\kappa_t(\tfrac{1}{2}\tau^2 - i\beta\tau)\} \tag{3.6}$$

with κ_t given by (3.5). On the other hand, in view of (2.6),

$$\varphi_t(\tau) = \exp[t\delta\{\sqrt{(\alpha^2 - \beta^2)} - \sqrt{(\alpha^2 - (\beta + i\tau)^2)}\}]. \tag{3.7}$$

Let $\kappa(v) = \kappa_1(v)$ and $\varphi(\tau) = \varphi_1(\tau)$. From Halgreen (1979) we have

$$\kappa(v) = -\delta^2 \int_{\psi/2}^{\infty} w(2\delta^2(y - \psi/2))^{-1} \log(1 + v/y)\, dy \tag{3.8}$$

where the function w is given by

$$w(x) = (\pi^2/2)x\{J_{1/2}^2(\sqrt{x}) + Y_{1/2}^2(\sqrt{x})\}. \tag{3.9}$$

Inserting the well-known expressions for the Bessel functions entering (3.9)

$$J_{1/2}(x) = \sqrt{(2/\pi)}x^{-1/2}\sin x \quad Y_{1/2}(x) = -\sqrt{(2/\pi)}x^{-1/2}\cos x,$$

we obtain

$$w(x) = \pi\sqrt{x}. \tag{3.10}$$

It was furthermore shown in Halgreen (1979) that

$$-\log(1 + \tau^2/2 - i\beta\tau) = \int_{-\infty}^{\infty} \frac{\exp(i\tau x) - 1}{x^2} |x| \exp(-\sqrt{(2 + \beta^2)}|x| + \beta x)\, dx. \tag{3.11}$$

Combining (3.6), (3.8) and (3.11) we find that the Lévy–Khintchine formula for φ is

$$\log\varphi(\tau) = i\tau\chi + \int_{|x|<1} (\exp(i\tau x) - 1 - i\tau x) f(x; \alpha, \beta, \delta)\, dx + \int_{|x|\geq 1} (\exp(i\tau x) - 1) f(x; \alpha, \beta, \delta)\, dx \tag{3.12}$$

where

$$\chi = 2\pi^{-1}\delta\alpha \int_0^1 \sinh(\beta x) K_1(\alpha x)\, dx \tag{3.13}$$

and where the Lévy density $f(x; \alpha, \beta, \delta)$ is given by

$$f(x; \alpha, \beta, \delta) = (\pi\sqrt{2})^{-1}\delta|x|^{-1} \exp(\beta x) \int_0^\infty y^{-1/2} \exp(-|x|\sqrt{(2y+\alpha^2)})\, dy$$

$$= \pi^{-1}\sqrt{2}\delta|x|^{-1} \exp(\beta x) \int_0^\infty \exp(-|x|\sqrt{2}\sqrt{(\alpha^2/2+s^2)})\, ds \quad (3.14)$$

or, equivalently,

$$f(x; \alpha, \beta, \delta) = \pi^{-1}\delta\alpha|x|^{-1} \exp(\beta x) K_1(\alpha|x|). \quad (3.15)$$

(The integral representation for the Bessel function K_1, used in deriving (3.14) and (3.15), is implicit in (4.1) for the density of the hyperbolic distribution given below.)

Equations (3.14) and (3.15) show that the normal inverse Gaussian Lévy process is representable in the form

$$x_t = t\chi + \int_{|y|<1} y\{n_t(dy) - t\nu(dy)\} + \int_{|y|\geq 1} y n_t(dy) \quad (3.16)$$

where

$$\nu(dy) = f(y; \alpha, \beta, \delta)\, dy; \quad (3.17)$$

furthermore, for any set Λ such that $0 \notin \bar{\Lambda}$ we have that $n_t^\Lambda = \int_\Lambda n_t(dy)$ is a Poisson process with parameter $\nu(\Lambda)$, and n_t^Λ and n_t^Ξ are independent if Λ and Ξ are disjoint (cf. Protter, 1992, p. 32).

For $y \downarrow 0$ we have $K_1(y) \sim y^{-1}$ and hence

$$f(y; \alpha, \beta, \delta) \sim \pi^{-1}\delta y^{-2} \quad \text{as } y \to 0. \quad (3.18)$$

The small jumps are therefore dominating the behaviour of the process x_t, and x_t has infinite variation on any finite time interval, cf. Gikhman & Skorohod (1975; th. 8, p. 279).

4. Connection to generalized hyperbolic and inverse Gaussian distributions

The density (2.1) determines, in fact, one of the generalized hyperbolic distributions (Barndorff-Nielsen, 1977), the hyperbolic distribution itself, whose density is

$$h(x; \alpha, \beta, \mu, \delta) = \sqrt{(\alpha^2-\beta^2)}/\{2\delta\alpha K_1(\delta\sqrt{(\alpha^2-\beta^2)})\}$$
$$\times \exp[-\alpha\sqrt{\{\delta^2+(x-\mu)^2\}} + \beta(x-\mu)], \quad (4.1)$$

being another special case. In (4.1), α and β have to be restricted by $0 \leqslant |\beta| < \alpha$.

Each of the generalized hyperbolic distributions are characterized by four parameters, α, β, μ and δ. Of these, μ and δ are location and scale parameters, and the alternative parameters $\bar{\alpha} = \delta\alpha$ and $\bar{\beta} = \delta\beta$ are invariant under location-scale changes.

The generalized hyperbolic distributions are all representable as normal variance-mean mixtures with generalized inverse Gaussian distributions as mixing distributions (Barndorff-Nielsen, 1977, 1978; see also Barndorff-Nielsen *et al.*, 1982). We shall denote the generalized inverse Gaussian distribution by $GIG(\lambda, \delta, \gamma)$. It has density function

$$(\gamma/\delta)^\lambda\{2K_\lambda(\delta\gamma)\}^{-1}z^{\lambda-1} \exp\{-\tfrac{1}{2}(\delta^2 z^{-1} + \gamma^2 z)\}. \quad (4.2)$$

If z follows the $GIG(\lambda, \delta, \sqrt{(\alpha^2-\beta^2)})$ distribution and if x given z is $N(\mu+\beta z, z)$ distributed, then unconditionally x has the generalized hyperbolic distribution $H(\lambda, \alpha, \beta, \mu, \delta)$ with density

$$a(\lambda, \alpha, \beta, \mu, \delta) q\left(\frac{x-\mu}{\delta}\right)^{\lambda-1/2} K_{\lambda-1/2}\left(\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right) \exp(\beta x) \tag{4.3}$$

where q is defined by (2.3) and

$$a(\lambda, \alpha, \beta, \mu, \delta) = (2\pi)^{-1/2}\delta^{-1/2}\alpha^{-\lambda+1/2}(\alpha^2-\beta^2)^{\lambda/2}K_\lambda(\delta\sqrt{(\alpha^2-\beta^2)})^{-1}\exp(-\beta\mu). \tag{4.4}$$

The normal inverse Gaussian distribution and the hyperbolic distribution correspond to the values $\lambda = -\frac{1}{2}$ and $\lambda = 1$, respectively. Other cases of special interest are $\lambda = 0$ (hyperboloid distribution) and $\lambda = \frac{1}{2}$. A computer program for maximum likelihood estimation based on i.i.d. observations from any of these special distributions, as well as for certain two- and three-dimensional generalizations, has been developed by Blæsild & Sørensen (1992, 1996).

For later reference we note that the conditional distribution of the mixing variable z given x is generalized inverse Gaussian:

$$z \mid x \sim GIG\left(\lambda - \frac{1}{2}, \delta q\left(\frac{x-\mu}{\delta}\right), \alpha\right) \tag{4.5}$$

Note, in particular, that this conditional distribution does not depend on β.

5. Comparison to hyperbolic modelling

Statistical modelling by means of the hyperbolic distribution has been effective in a number of contexts, see for instance Barndorff-Nielsen (1977, 1986), Blæsild (1981), Barndorff-Nielsen & Blæsild (1983), Barndorff-Nielsen *et al.* (1985), Barndorff-Nielsen & Christiansen (1988) and further references given below.

The normal inverse Gaussian distribution can approximate most hyperbolic distributions very closely but can also describe observations with considerably heavier tail behaviour than the log linear rate of decrease that characterizes the hyperbolic shape. Since, in addition, the normal inverse Gaussian distribution has more tractable probabilistic properties than the hyperbolic it seems potentially of substantial usefulness. In the rest of this section and part of the following we consider this point further, in two particular contexts: turbulence and finance.

5.1. Turbulence

The study of velocity differences in moderate and high Reynolds number turbulent wind fields is of central importance in turbulence, both theoretically and practically. Numerous and extensive observational investigations have shown that the velocity differences typically follow distributions that are close to symmetric and have tails that are either nearly log linear or somewhat heavier than log linear, cf. for instance van Atta & Park (1972), Wyngaard & Tennekes (1970) and Wyngaard & Pao (1972). The log linearity has motivated a number of studies (Barndorff-Nielsen, 1979, 1986; Barndorff-Nielsen *et al.*, 1989, 1990, 1993) relating to questions in turbulence and evolving from the hyperbolic distribution.

However, the normal inverse Gaussian distribution seems to offer an attractive alternative starting point for parametric modelling in turbulence because of its special probabilistic properties and its ability to describe the typical tail behaviour of the velocity differences.

This possibility will not be pursued specifically here. Instead we turn to the modelling of returns from financial assets such as stocks or currencies.

5.2. Finance

Recent investigations, by Eberlein & Keller (1995) and Küchler *et al.* (1994), have demonstrated that the hyperbolic distribution provides a very good fit to the distributions of daily returns, measured on the log scale, of single stocks or portfolios of stocks from a number of leading German enterprises. The time series of daily returns concerned do not exhibit significant autocorrelations, nor do the derived series of squared returns. It is therefore natural to try to model the logarithmic stock price processes as Lévy processes, and Eberlein & Keller (1995) introduce and study what they term the hyperbolic Lévy process for this purpose, using the fact that the hyperbolic distribution is infinitely divisible.

In a preliminary version of the present report (Barndorff-Nielsen, 1995) it was demonstrated graphically that the normal inverse Gaussian distribution can approximate most hyperbolic distributions extremely closely, including those found by Eberlein & Keller (1995). Later work by Blæsild (1995) and Rydberg (1996) has shown that the normal inverse Gaussian distribution provides an even better description of the German data than the hyperbolic, and that the data point to the normal inverse Gaussian distribution as being the most appropriate within the class of generalized hyperbolic distributions. In addition, the normal inverse Gaussian Lévy process is, as already discussed, mathematically simpler than the hyperbolic Lévy process.

It is, moreover, rather typical that asset returns exhibit tail behaviour that is somewhat heavier than log linear, and this further strengthens the case for the normal inverse Gaussian in the financial context.

6. Conditional heteroscedasticity and stochastic volatility modelling

The models for daily German stock returns discussed in the previous section were i.i.d. hyperbolic and i.i.d. normal inverse Gaussian. However, logarithmic stock returns and other types of financial time series often exhibit a significant dependence structure, cf. for instance, Shephard (1995). This dependence structure is of a nature generally referred to by the terms "stochastic volatility and conditional heteroscedasticity". Remarkably, a similar type of dependence is a characteristic feature in turbulence studies where it is referred to as "intermittency".

At the end of the present section we shall define extensions of the normal inverse Gaussian i.i.d. model to time series with dependencies of the type in question.

Prior to this we discuss an observation-driven approach to state space modelling, in discrete time. (We use the term "observation-driven" in the sense introduced by Cox, 1981.) This will then be specialized to the financial setting. For an overview of related work on state space modelling in finance see Shephard (1995).

6.1. Observation-driven state space modelling

For brevity we consider only the simplest possible case.

Let $y_0, z_1, y_1, z_2, y_2, \ldots$ be a Markov chain of continuous type random variates. We consider z_i, $i = 1, 2, \ldots$, to be a sequence of unknown states while $y_j, j = 0, 1, 2, \ldots$, are observable. The joint law of $\{z_i\}$ and $\{y_j\}$ is specified uniquely by prescribing the transition probability densities $p(z_i \mid y_{i-1})$ and $p(y_i \mid z_i)$, $i = 1, 2, \ldots$, and the observable process $\{y_j\}$

constitutes a Markov chain whose transition probabilities are expressible in terms of the known densities $p(z_i \mid y_{i-1})$ and $p(y_i \mid z_i)$ by the relation

$$p(y_i \mid y_{i-1}) = \int p(y_i \mid z_i) p(z_i \mid y_{i-1})\, dz_i. \tag{6.1}$$

In parametric statistical modelling $p(y_i \mid y_{i-1})$ will depend on a parameter ω, some components of which may come from $p(z_i \mid y_{i-1})$, others from $p(y_i \mid z_i)$. In the particular cases to be discussed below, the integral in (6.1) can be explicitly calculated and hence an explicit expression for the likelihood function for ω based on data $y^n \doteq (y_0, \ldots, y_n)$

$$L(\omega) = \prod_{i=1}^{n} p(y_i; \omega \mid y_{i-1}) \tag{6.2}$$

is available. Note also that we then have an explicit expression for the conditional density $p(z_1, \ldots, z_n \mid y_0, y_1, \ldots, y_n)$ of the states $z_1, \ldots, z_n$ and, in particular, for $p(z_n \mid y_0, y_1, \ldots, y_n)$, the latter being given by

$$p(z_n \mid y_0, y_1, \ldots, y_n) = p(z_n \mid y_{n-1}, y_n) = p(z_n \mid y_{n-1}) p(y_n \mid z_n) / p(y_n \mid y_{n-1}). \tag{6.3}$$

Consider now the special case that $p(y_i \mid z_i)$ is, for all i, the density of the normal distribution $N(\mu + \beta z_i, z_i)$, and let $p(z_i \mid y_{i-1})$ be given by the generalized inverse Gaussian distribution

$$z_i \mid y_{i-1} \sim GIG(\lambda, r(y_{i-1}; \eta), \gamma) \tag{6.4}$$

where $r(\cdot\,; \eta)$ is some positive function, depending on a parameter η. The precise form of $r(\cdot\,; \eta)$ may be chosen based on theoretical and empirical knowledge. Then, by (6.1),

$$y_i \mid y_{i-1} \sim H(\lambda, \alpha, \beta, \mu, r(y_{i-1}; \eta)) \tag{6.5}$$

with $\alpha = \sqrt{(\beta^2 + \gamma^2)}$, cf. section 4. Note also that, writing r_i for $r(y_i; \eta)$, we have

$$z_i \mid y^n \sim z_i \mid (y_{i-1}, y_i) \sim GIG(\lambda - \tfrac{1}{2}, \sqrt{\{r_{i-1}^2 + (y_i - \mu)^2\}}, \alpha), \tag{6.6}$$

cf. (4.5) and (6.3).

The relation between the dependence structure of $\{z_i\}$ and that of $\{y_j\}$ is of interest. Suppose for simplicity that $\mu = \beta = 0$. In view of (6.4) we have

$$\begin{aligned} E(z_i \mid z_{i-1}) &= E\{E(z_i \mid z_{i-1}, y_{i-1}) \mid z_{i-1}\} = E\{E(z_i \mid y_{i-1}) \mid z_{i-1}\} \\ &= \alpha^{-1} E\left\{ r(y_{i-1}; \eta) \frac{K_{\lambda+1}(\alpha r(y_{i-1}; \eta))}{K_\lambda(\alpha r(y_{i-1}; \eta))} \mid z_{i-1} \right\}. \end{aligned} \tag{6.7}$$

If $\lambda = -\frac{1}{2}$, as we shall assume from now on, (6.7) reduces to

$$E(z_i \mid z_{i-1}) = \alpha^{-1} E\{r(y_{i-1}; \eta) \mid z_{i-1}\}, \tag{6.8}$$

and in this case the conditional variance of z_i given z_{i-1} satisfies

$$V(z_i \mid z_{i-1}) = \alpha^{-2} [E(z_i \mid z_{i-1}) + V\{r(y_{i-1}; \eta) \mid z_{i-1}\}]. \tag{6.9}$$

Further, since $y_i \mid y_{i-1} \sim NIG(\alpha, 0, 0, r(y_{i-1}; \eta))$ we have

$$E(y_i^2 \mid y_{i-1}) = E(y_i^2 \mid y_{i-1}^2) = \alpha^{-1} r(y_{i-1}; \eta) \tag{6.10}$$

cf. (2.8). A plot of y_i^2 against y_{i-1}, $i = 1, 2, \ldots$, can thus indicate an appropriate formula for $r(\cdot\,; \eta)$.

Suppose a functional form of $r(\cdot\,;\eta)$ has been chosen. To determine the maximum likelihood estimates of the parameters α, β, μ and η, based on a sample $y_0, y_1, \ldots, y_n$, it may be convenient, rather than seeking to maximize the likelihood function simultaneously in all the parameters, to proceed via determination of the profile likelihood for η using the previously mentioned computer program developed by Blæsild & Sørensen (1996) to find the partial maximum likelihood estimates of α, β and μ (which would involve treating the quantities $y_j/r(y_{j-1};\eta)$, $j=0, 1, \ldots, n$, as if they were normal inverse Gaussian i.i.d. with $\delta = 1$).

A rather flexible choice of $r(\cdot\,;\eta)$ is, with $\eta = (\varepsilon, \rho, \kappa)$,

$$r(y;\eta) = (\varepsilon + \rho y^2)^\kappa. \tag{6.11}$$

For $\kappa = 0$ we recover the normal inverse Gaussian i.i.d. model and for general $\kappa \geqslant 0$ the expression (6.8) for $E(z_i \mid z_{i-1})$ can be approximately evaluated as

$$E(z_i \mid z_{i-1}) = \alpha^{-1}\{\varepsilon + 2\pi^{-1/2}\Gamma(\kappa + \tfrac{1}{2})\rho z_{i-1}\}^\kappa \tag{6.12}$$

where Γ denotes the gamma function. In fact, the formula is exact both for $\kappa = 0$ and $\kappa = 1$, interpolates smoothly between these two cases, and holds asymptotically for $z_{i-1} \to 0$ and $z_{i-1} \to \infty$ whatever the value of κ.

Thus, if for suitable choice of ε, ρ and κ, determined for instance via the profile likelihood for $\eta = (\varepsilon, \rho, \kappa)$, we have that the model fits the data $y_0, \ldots, y_n$ well then by (6.12) we have a direct handle on the regression structure of the $\{z_i\}$ process.

In the language of financial modelling, the $z_i^{1/2}$ are referred to as the stochastic volatilities. In that context it is often considered natural to suppose that the volatility process is "mean reverting" in the sense that if z_{i-1} is below the "typical" level of the process then the next value z_i will tend to be larger than z_{i-1}, and conversely if z_{i-1} is larger than the typical value. If $0 < \kappa < 1$ then (6.11) entails this kind of behaviour.

In view of (6.6), an appealing choice is $\kappa = \frac{1}{2}$, in which case (6.11) and (6.6) (with $\lambda = -\frac{1}{2}$) take the form

$$r(y;\varepsilon,\rho) = (\varepsilon + \rho y^2)^{1/2} \tag{6.13}$$

respectively

$$z_i \mid y^n \sim GIG(-1, \sqrt{(\varepsilon + \rho y_{i-1}^2 + y_i^2)}, \alpha). \tag{6.14}$$

The conditional mean of z_i, which may be used as a predictor of z_i, is then

$$E(z_i \mid y^n) = \alpha^{-1}\frac{K_0(\alpha\sqrt{(\varepsilon + \rho y_{i-1}^2 + y_i^2)})}{K_1(\alpha\sqrt{(\varepsilon + \rho y_{i-1}^2 + y_i^2)})}. \tag{6.15}$$

Extensions of the above approach in order to model higher order dependence is rather straightforward. For instance, the law of the process $y_0, z_1, y_1, z_2, \ldots,$ may be specified by saying that

$$y_i \mid (z^i, y^{i-1}) \sim y_i \mid z_i \sim N(0, z_i) \tag{6.16}$$

(where $z^i = (z_1, \ldots, z_i)$) and that

$$z_i \mid y^{i-1} \sim IG((\varepsilon + \rho_1 y_{i-1}^2 + \cdots + \rho_k y_{i-k}^2)^\kappa, \alpha) \tag{6.17}$$

for some positive integer k and letting, say, $y_{-k+1} = \cdots = y_0 = 0$. The parameters $\varepsilon, \rho_1, \ldots, \rho_k$ and κ are supposed to be non-negative.

Noting that

$$E(z_i \mid y^{i-1}) = \alpha^{-1}(\varepsilon + \rho_1 y_{i-1}^2 + \cdots + \rho_k y_{i-k}^2)^\kappa \tag{6.18}$$

one sees that the present type of model has a certain analogy to ARCH models (cf. for instance, Shephard, 1995). However, in contrast to the latter, the conditional law of the volatility at time t given the observations from the previous time points is here non-degenerate.

Under the model given by (6.17) with $\kappa = \frac{1}{2}$ we have moreover

$$y_i \mid y^{i-1} \sim NIG(\alpha, 0, 0, (\varepsilon + \rho_1 y_{i-1}^2 + \cdots + \rho_k y_{i-k}^2)^{1/2}) \tag{6.19}$$

so that, again, the likelihood function is explicitly available and the maximum likelihood estimation may be carried out in analogy with the special case $k = 1$ considered earlier. Furthermore,

$$E(y_i^2 \mid y^{i-1}) = \alpha^{-1}(\varepsilon + \rho_1 y_{i-1}^2 + \cdots + \rho_k y_{i-k}^2)^{1/2} \tag{6.20}$$

and

$$z_i \mid y^n \sim GIG(-1, (\varepsilon + \rho_1 y_{i-1}^2 + \cdots + \rho_k y_{i-k}^2)^{1/2}, \alpha). \tag{6.21}$$

6.2. Stochastic volatility models

Typically, the autocorrelations of an observed series y_j of financial asset returns are essentially 0 whereas the squared series y_j^2 have positive autocorrelations that decrease slowly to 0. An appealing way to model this is to specify y_j as

$$y_j = \sigma_j \varepsilon_j \tag{6.22}$$

where $\{\sigma_j\}$ and $\{\varepsilon_j\}$ are independent processes with the σ_j being positive, generally dependent, random variables and the ε_j being i.i.d. normal variates. In the mathematical finance terminology, the variables σ_j are called stochastic volatilities. For a recent review of stochastic volatility models of the type (6.22) see Shephard (1995).

If the y_j are observations of the normal inverse Gaussian processes discussed in subsection 6.1 above and if $\mu = \beta = 0$ then the y_j are of type (6.22), with z_j in the role of σ_j^2.

7. Multivariate versions

A multivariate extension of the generalized hyperbolic distributions, and in particular of the hyperbolic distribution itself and of the normal inverse Gaussian distribution, is simple to construct in view of the normal mixture representation, cf. Barndorff-Nielsen (1977). Suppose that x is a random vector which, conditional on a random variable z, follows the m-dimensional normal distribution with mean vector and variance matrix of the form $\mu + z\beta\Delta$ and $z\Delta$, respectively; here, for identifiability of the parameters we assume that $|\Delta| = 1$. Then, if again z is distributed according to the generalized inverse Gaussian distribution $GIG(\lambda, \delta, \gamma)$ (cf. (4.2)) where

$$\gamma = \sqrt{(\alpha^2 - \beta\Delta\beta^{\mathrm{T}})} \tag{7.1}$$

we have as an m-dimensional generalization of (4.3–4)

$$a(\lambda, \alpha, \beta, \mu, \delta, \Delta) Q^{\lambda - 1/2} K_{\lambda - 1/2}(\delta\alpha Q) \exp(\beta x^{\mathrm{T}}) \tag{7.2}$$

where, with $\Upsilon = \delta\Delta$ and q defined by (2.3),

$$Q = q((x - \mu)\Upsilon^{-1}(x - \mu)^{\mathrm{T}}) \tag{7.3}$$

and

$$a(\lambda, \alpha, \beta, \mu, \delta, \Delta) = (2\pi)^{-m/2}\delta^{-(m+\lambda)/2}\gamma^{\lambda}\alpha^{m/2-\lambda}K_{\lambda}(\delta\gamma)^{-1}\exp(-\beta\mu^{\mathrm{T}}). \tag{7.4}$$

We denote this distribution by $H_m(\lambda, \alpha, \beta, \mu, \delta, \Delta)$. The class of multivariate generalized hyperbolic distributions is closed under conditioning, marginalization and affine transformations. For discussions of these and other properties and applications of the distributions, see Blæsild (1981) and Blæsild & Jensen (1981).

Setting $\lambda = -\frac{1}{2}$ in (7.2) we obtain what we shall term the m-dimensional normal inverse Gaussian distribution $NIG_m(\alpha, \beta, \mu, \delta, \Delta)$, which has density

$$g(x; \alpha, \beta, \mu, \delta, \Delta) = a(\alpha, \beta, \mu, \delta, \Delta)b(x; \alpha, \mu, \delta, \Delta)\exp(\beta x^{\mathrm{T}}) \tag{7.5}$$

where

$$a(\alpha, \beta, \mu, \delta, \Delta) = \{(2\delta)^{m-1}\pi^{m+1}|\Delta|\}^{-1/2}\alpha^{(m+1)/2}\exp(\delta\gamma)\exp(-\beta\mu^{\mathrm{T}}) \tag{7.6}$$

and

$$b(x; \alpha, \mu, \delta, \Delta) = q((x-\mu)\Upsilon^{-1}(x-\mu)^{\mathrm{T}})^{-(m+1)/2}K_{(m+1)/2}(\delta\alpha q((x-\mu)\Upsilon^{-1}(x-\mu)^{\mathrm{T}})). \tag{7.7}$$

For m even, say $m = 2k$, the expression for b, and hence for g, has a more explicit form because in that case the Bessel function $K_{(m+1)/2} = K_{k+1/2}$ satisfies

$$K_{k+1/2}(s) = \sqrt{(\pi/2)}s^{-1/2}\exp(-s)\left\{1 + \sum_{i=1}^{k}\frac{(k+i)!}{(k-i)!i!}(2s)^{-i}\right\}. \tag{7.8}$$

The joint distribution of z and x has been studied recently, from entirely different viewpoints, by Hassaïri (1992, 1993) and Letac & Seshadri (1995). These authors refer to the distribution as "the inverse Gaussian distribution on R^{m+1}".

From the exponential model form of (7.5) we have, by (7.6), that the cumulant generating function of the $NIG_m(\alpha, \beta, \mu, \delta, \Delta)$ distribution is

$$K(u; \alpha, \beta, \mu, \delta, \Delta) = \delta\gamma - \delta\{\alpha^2 - (\beta+u)\Delta(\beta+u)^{\mathrm{T}}\}^{1/2} + \mu u^{\mathrm{T}}. \tag{7.9}$$

Consequently, if $x \sim NIG_m(\alpha, \beta, \mu, \delta, \Delta)$ then the mean and variance of x are

$$Ex = \mu + (\delta/\gamma)\beta\Delta \tag{7.10}$$

$$Vx = (\delta/\gamma^3)(\gamma^2\Delta + \Delta^{\mathrm{T}}\beta^{\mathrm{T}}\beta\Delta). \tag{7.11}$$

For $\mu = \beta = 0$ these formulae reduce to $Ex = 0$ and

$$Vx = (\delta/\alpha)\Delta. \tag{7.12}$$

In particular, then the correlation matrix of x is the same as under the conditional distribution of x given z, a property of direct interest for financial modelling.

The multivariate Lévy process corresponding to the normal inverse Gaussian distribution (7.5) is obtainable, in extension of (3.3), by subordination of a multivariate Brownian motion relative to the inverse Gaussian Lévy process $\{z_t\}$, followed by an affine transformation. Specifically, let $\{b_t\}$ denote Brownian motion with drift of the form βB, where B is an $m \times m$ matrix with $BB^{\mathrm{T}} = \Delta$, and let $\{z_t\}$ be the inverse Gaussian homogeneous Lévy process with $z_t \sim IG(t\delta, \gamma)$. Then the m-dimensional normal inverse Gaussian Lévy process, for which $x_t \sim NIG_m(\alpha, \beta, \mu, \delta, \Delta)$, is representable as

$$x_t = b_{z_t}B^{\mathrm{T}} + t\mu. \tag{7.13}$$

The state space and stochastic volatility models considered in section 6 may also be generalized to multivariate settings.

Suppose for instance that $y_0, z_1, y_1, z_2, \ldots, y_{i-1}, z_i, y_i, \ldots$ is Markovian with

$$z_i \mid y_{i-1} \sim IG(r(y_{i-1}; \eta), \alpha), \quad (7.14)$$

and

$$y_i \mid z_i \sim N(0, z_i \Delta); \quad (7.15)$$

here $r(\cdot\,; \eta)$, to be specified, is some positive function defined on R^m and depending on a parameter η, and Δ is a positive definite $m \times m$ matrix with $|\Delta| = 1$. Then

$$y_i \mid y_{i-1} \sim NIG_m(\alpha, 0, 0, r(y_{i-1}; \eta), \Delta) \quad (7.16)$$

and hence, in particular, the likelihood function for (α, Δ, η) is readily available.

In analogy with (6.13) we may let

$$r(y; \eta) = r(y; \varepsilon, R) = (\varepsilon + yRy^{\mathrm{T}})^{1/2} \quad (7.17)$$

where R is a positive definite $m \times m$ matrix. In a financial context, the elements of R determine how the various stocks, expressed through the coordinates of y_{i-1}, singly and as pairs influence the volatility at the next observation time i.

With the present choice of the function r we have

$$z_i \mid y^n \sim GIG\left(-\frac{m+1}{2}, (\varepsilon + y_{i-1} R y_{i-1}^{\mathrm{T}} + y_i \Delta^{-1} y_i^{\mathrm{T}})^{1/2}, \alpha\right). \quad (7.18)$$

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